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TOPOLOGICAL GROUPS WITH MANY SMALL SUBGROUPS REVISITED

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As usual, \mathbb{Z} and \mathbb{Q} denote the groups of integer numbers and rational numbers respectively, \mathbb{N} denotes the set of natural numbers and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$.

Definition 0.1. For a subset A of a topological group G , we denote by $\langle A \rangle$ the smallest subgroup of G containing A , and we define

$$\langle A \rangle_k = \left\{ \prod_{i=1}^j a_i : j \leq k, a_1, \dots, a_j \in A \right\}$$

for every $k \in \mathbb{N}^+$.

Remark 0.2. If A is a subset of a group G such that $A = A^{-1}$, then $\langle A \rangle = \bigcup_{k \in \mathbb{N}^+} \langle A \rangle_k$.

1. THE SMALL SUBGROUP GENERATING PROPERTY

In the realm of topological groups von Neumann [11] introduced the class of the *minimally almost periodic groups*. A topological group G satisfies this property if every continuous homomorphism of G to a compact group K is trivial. Examples of minimally almost periodic groups are rather difficult or cumbersome to find; see [5] for the overview of the difficulties involved. Therefore, one approach towards simplifying the search for examples can be done by considering a narrower class of groups such that every group in this class can be easily checked to have the minimal almost periodic property. This was done by Gould in [9] who introduced the *small subgroup generating property* (SSGP) while attempting to study the behavior of well-known examples of minimally almost periodic groups.

Definition 1.1. A topological group G has the *small subgroup generating property*, abbreviated to *SSGP*, provided that for every neighbourhood U of the identity of G , there exists a family \mathcal{H} of subgroups of G such that $\bigcup \mathcal{H} \subseteq U$ and the smallest closed subgroup N of G containing $\bigcup \mathcal{H}$ coincides with G .

One can easily see that SSGP groups are minimally almost periodic:

Proposition 1.2. [1] *If G is an SSGP group, then G is minimally almost periodic.*

Proof. Recall that every compact group is isomorphic to a subgroup of a product of finite-dimensional unitary groups. Thanks to this fact, it suffices to prove that every continuous homomorphism f of G to the n -dimensional unitary group $\mathbb{U}(n)$ is trivial. Recall that $\mathbb{U}(n)$ satisfies the *no small subgroups property*; that is, there exists a neighborhood U of e in $\mathbb{U}(n)$ that contains no non-trivial subgroups. Now $V = f^{-1}(U)$ is an open neighbourhood of e in G . Since G is SSGP, there exists a family \mathcal{H} of subgroups of G such that

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$\bigcup \mathcal{H} \subseteq V$ and G coincides with the smallest closed subgroup of G containing $\bigcup \mathcal{H}$. Since f is an homomorphism, it maps all the subgroups in \mathcal{H} into subgroups of $U(n)$. However, by definition, $f(\bigcup \mathcal{H}) \subseteq f(V) \subseteq U$, so all these subgroups in \mathcal{H} are mapped to the trivial group $\{e\}$. This implies $f(\bigcup \mathcal{H}) = \{e\}$. Since f is a homomorphism, $f(\langle \bigcup \mathcal{H} \rangle) = \{e\}$. Since G coincides with the closure of $\langle \bigcup \mathcal{H} \rangle$ and f is continuous, G is mapped to the closure of $\{e\}$. Therefore, f is a trivial mapping and the result follows. \square

Comfort and Gould established the following fundamental fact:

Theorem 1.3. [1, Corollary 2.23] *A bounded torsion abelian topological group is SSGP if and only if it is minimally almost periodic.*

The following explicit examples of abelian groups admitting an SSGP group topology were given [1]:

Theorem 1.4. *The following abelian groups admit an SSGP topology:*

- (a) *all subgroups \mathbb{Q}_π of \mathbb{Q} with infinite π (in particular, \mathbb{Q} itself);*
- (b) *$\mathbb{Q}_\pi/\mathbb{Z}$ where π is an infinite set of primes (in particular, \mathbb{Q}/\mathbb{Z});¹*
- (c) *direct sums of the form $\oplus \mathbb{Z}(p_i)$ where the primes p_i all coincide or differ;*
- (d) *$\mathbb{Z}^{(\omega)}$ (the direct sum);*
- (e) *\mathbb{Z}^ω (the full product);*
- (f) *$G^{(\lambda)}$ for $|G| > 1$ and $\lambda \geq \omega$;*
- (g) *F^λ for $1 < |F| < \omega$ and $\lambda \geq \omega$;*
- (h) *arbitrary sums and products of groups which admit an SSGP group topology.*

Finally, Comfort and Gould [1] showed that two classical groups do not admit an SSGP group topology, while they both admit a minimally almost periodic group topology; see [5].

Example 1.5. [1] The group \mathbb{Z} of integer numbers and the Prüfer group $\mathbb{Z}(p^\infty)$ do not admit an SSGP group topology.

Following [4], for every subset A of a group G , define

$$(1) \quad \text{Cyc}(A) = \{x \in G : \langle x \rangle \subseteq A\},$$

where $\langle x \rangle$ denotes the smallest subgroup of G containing x , i.e., the cyclic subgroup generated by x .

Using this notation, a simple reformulation of the SSGP property for abelian groups can be given.

Proposition 1.6. [4] *An abelian topological group G has the small subgroup generating property if and only if $\langle \text{Cyc}(U) \rangle$ is dense in G for every neighbourhood U of zero of G .*

A more algebraic reformulation is given in the next proposition which was proved in [12] for abelian groups G .

Proposition 1.7. *A topological group G has the small subgroup generating property if and only if*

$$G = \bigcup_{k \in \mathbb{N}^+} \langle \text{Cyc}(W) \rangle_k W$$

for every neighbourhood W of e_G in G .

Comfort and Gould [1] asked the following question.

Question 1.8. [1, Question 5.2] *What are the (abelian) groups which admit an SSGP topology?*

¹We refer the reader to Definition 4.7 for the notation \mathbb{Q}_π appearing in items (a) and (b).

2. THE CLASSES $\text{SSGP}(n)$

An infinite sequence of proper subclasses of the class of minimally periodic groups was defined in [1].

Definition 2.1. Let G be a topological group.

- (a) G has $\text{SSGP}(0)$ if G is the trivial group.
- (b) For $n \in \mathbb{N}^+$, G has $\text{SSGP}(n)$ provided that, for every neighbourhood U of the identity of G , there exists a family \mathcal{H} of subgroups of G such that $\bigcup \mathcal{H} \subseteq U$ and the smallest closed subgroup N of G containing $\bigcup \mathcal{H}$ is normal in G and G/N has $\text{SSGP}(n-1)$.

One can easily see that the class $\text{SSGP}(1)$ coincides with the class of minimally almost periodic groups.

It was proved in [1, Remark 3.4, Theorem 3.5] that

$$(2) \quad \text{SSGP} = \text{SSGP}(1) \rightarrow \text{SSGP}(2) \rightarrow \dots \rightarrow \text{SSGP}(n) \rightarrow \text{SSGP}(n+1) \rightarrow \dots \rightarrow \text{minimally almost periodic}.$$

Examples distinguishing all classes in (2) can be found in [1, Corollary 3.14, Theorem 4.6].

The following two examples are of particular interest:

Example 2.2. [1] The group \mathbb{Z} of integer numbers and the Prüfer group $\mathbb{Z}(p^\infty)$ admit no $\text{SSGP}(n)$ topology for any $n \in \mathbb{N}^+$.

Comfort and Gould [1] asked the following question.

Question 2.3. [1, Question 5.3] Does every abelian group which for some $n > 1$ admits an $\text{SSGP}(n)$ topology also admit an SSGP topology?

3. THE CLASSES $\text{SSGP}(\alpha)$

In [4], Dikranjan and the first listed author utilized an operator based approach to further extend the classes $\text{SSGP}(n)$ to any ordinal α . We outline this approach here.

Definition 3.1. Let G be a topological group, A be a subset of G and H be a subgroup of G .

- (i) $N(H)$ denotes the maximal normal subgroup of G contained in H .²
- (ii) $Cs(A)$ is the smallest closed subgroup of G containing A .

Note that $N(H)$ is closed in G whenever H is closed in G , as the closure of a normal subgroup of G is a normal subgroup of G .

The operator Cyc defined in (1) behaves as an interior operator on G and Cs behaves as a closure operator on G [4].

Let us consider $\mathbf{S} = N \circ Cs \circ \text{Cyc}$ as the composition of the operators Cyc , Cs and N ; that is, $\mathbf{S}(X) = N(Cs(\text{Cyc}(X)))$ for every $X \subseteq G$.

By transfinite induction, for every ordinal α , define the α 's iteration $\mathbf{S}^{(\alpha)}$ of \mathbf{S} as follows. Let $\mathbf{S}^{(0)}(X) = \{e_G\}$ for every $X \subseteq G$. If $\alpha > 0$ is an ordinal and $\mathbf{S}^{(\beta)}(X)$ has already

²Its existence follows from the fact that the family \mathcal{N} of normal subgroups of G contained in H is directed, as the product $N_1 N_2$ of two members of \mathcal{N} still belongs to \mathcal{N} , so $N(H) = \bigcup \mathcal{N}$.

been defined for all $\beta < \alpha$, let

$$\mathbf{S}^{(\alpha)}(X) = \mathbf{S} \left(X \cdot \bigcup_{\beta < \alpha} \mathbf{S}^{(\beta)}(X) \right) \quad \text{for every } X \subseteq G.$$

Definition 3.2. [4, Definition 2.3] For an ordinal α , a topological group G is said to be an $\text{SSGP}(\alpha)$ group provided that $\mathbf{S}^{(\alpha)}(U) = G$ for every neighbourhood U of the identity of G .

The connection of this definition with Definition 2.1 is seen from the following theorem.

Theorem 3.3. [4, Theorem 6.4] *Let $n \in \mathbb{N}^+$. An abelian topological group is an $\text{SSGP}(n)$ group in the sense of Definition 2.1 if and only if it is an $\text{SSGP}(n)$ group in the sense of Definition 3.2.*

This theorem shows that the sequence of properties $\text{SSGP}(n)$ is a natural extension for the $\text{SSGP}(n)$ properties in the realm of Abelian groups. Do note that the use of quotients is replaced with a description that depends only on the operator \mathbf{S} .

Now, if α and β are ordinals with $\beta < \alpha$, then $\text{SSGP}(\beta) \rightarrow \text{SSGP}(\alpha)$ [4, Proposition 5.1]. In addition, every $\text{SSGP}(\alpha)$ group is minimally almost periodic [4, Proposition 5.3 (iii)]. Thus, we have the following natural extension of inclusions from (2).

$$(3) \quad \text{SSGP} = \text{SSGP}(1) \rightarrow \text{SSGP}(2) \rightarrow \dots \rightarrow \text{SSGP}(\alpha) \rightarrow \dots \rightarrow \text{SSGP}(\alpha + 1) \rightarrow \dots \rightarrow \text{minimally almost periodic}.$$

It follows from Theorem 1.3 that all the classes in (3) coincide for bounded torsion abelian topological groups.

4. EXISTENCE OF SSGP TOPOLOGIES IN THE ABELIAN CASE

Following [6], for an abelian group G , we denote by $r_0(G)$ the free rank of G , by $r_p(G)$ the p -rank of G , and we let

$$r(G) = \max \left\{ r_0(G), \sum \{ r_p(G) : p \in \mathbb{P} \} \right\},$$

where \mathbb{P} denotes the set of prime numbers.

Definition 4.1. [3, Definition 7.2] For an abelian group G , the cardinal

$$(4) \quad r_d(G) = \min \{ r(nG) : n \in \mathbb{N}^+ \}$$

is called the *divisible rank* of G .

The notion of the divisible rank was defined, under the name of *final rank*, by Szele [13] for p -groups.

Remark 4.2. An abelian group G satisfies $r_d(G) = 0$ if and only if G is a bounded torsion group; that is, if $nG = \{0\}$ for some $n \in \mathbb{N}^+$.

Recall that a non-trivial bounded torsion abelian group G is a direct sum

$$(5) \quad G = \bigoplus_{p \in \pi(G)} \bigoplus_{i=1}^{m_p} \mathbb{Z}(p^i)^{(\alpha_{p,i})}$$

of cyclic groups, where $\pi(G)$ is a non-empty finite set of prime numbers and the cardinals $\alpha_{p,i}$ are known as *Ulm-Kaplanski invariants* of G . Note that while some of them may be

equal to zero, the cardinals α_{p,m_p} must be positive; they are called *leading Ulm-Kaplanski invariants* of G .

Gabrielyan [7] proved that a non-trivial bounded abelian group admits a minimally almost periodic group topology precisely when all its leading Ulm-Kaplanski invariants are infinite. (An alternative proof of this result can be found also in [5].) Combining this with Theorem 1.3 and Remark 4.2, we get the following corollary.

Corollary 4.3. *A non-trivial abelian group G satisfying $r_d(G) = 0$ admits an SSGP group topology if and only if all leading Ulm-Kaplanski invariants of G are infinite.*

Dikranjan and the first author completely resolved Question 1.8 for abelian groups of infinite divisible rank.

Theorem 4.4. [4, Theorem 3.2] *Every abelian group G satisfying $r_d(G) \geq \omega$ admits an SSGP group topology.*

In the remaining case $0 < r_d(G) < \omega$, Dikranjan and the first author found a necessary condition on G in order to admit an SSGP group topology.

Theorem 4.5. [4, Theorem 3.8] *Let G be an abelian $\text{SSGP}(\alpha)$ group for some ordinal α . If $1 \leq r_d(G) < \infty$, then the quotient $H = G/t(G)$ of G with respect to its torsion part*

$$t(G) = \{g \in G : ng = 0 \text{ for some } n \in \mathbb{N}^+\}$$

has finite rank $r_0(H)$ and $r(H/A) = \omega$ for some (equivalently, every) free subgroup A of H such that H/A is torsion.

Dikranjan and the first author asked if the necessary condition given in Theorem 4.5 is also sufficient for the existence of an SSGP group topology on G . Moreover, the same authors reduced this problem to the following question:

Question 4.6. [4, Question 13.1] *Let $m \in \mathbb{N}^+$ and*

$$G = G_0 \times \left(\bigoplus_{i=1}^k \mathbb{Z}(p_i^\infty) \right) \times F,$$

where F is a finite group, $k \in \mathbb{N}$, p_1, p_2, \dots, p_k are (not necessarily distinct) prime numbers, and G_0 is a subgroup of \mathbb{Q}^m containing \mathbb{Z}^m such that $G_0 \not\subseteq \mathbb{Q}_\pi^m$ for every finite set π of prime numbers. Is it true that G admits an SSGP group topology?

The notation \mathbb{Q}_π appearing in the above question is given in the next definition.

Definition 4.7. For a set π of prime numbers, we use \mathbb{Q}_π to denote the set of all rational numbers q whose irreducible representation $q = z/n$ with $z \in \mathbb{Z}$ and $n \in \mathbb{N}^+$ is such that all prime divisors of n belong to π .

Dikranjan and the first author provisionally provided a theorem completely characterizing abelian groups G admitting an SSGP group topology in the remaining open case $0 < r_d(G) < \omega$ provided that the answer to Question 4.6 is positive [4, Theorem 13.2].

5. OUR RESULTS IN THE ABELIAN CASE

We provide a positive answer to a much more general version of Question 4.6:

Theorem 5.1. [12, Theorem 2.10] *Suppose that $m \in \mathbb{N}^+$ and G_0 is a subgroup of \mathbb{Q}^m containing \mathbb{Z}^m such that $G_0 \not\subseteq \mathbb{Q}_\pi^m$ for every finite set π of prime numbers. Then for each at most countable abelian group H , the product $G = G_0 \times H$ admits a (separable) metric SSGP group topology.*

Having validated a positive answer to Question 4.6, the final statement of [4, Theorem 13.2] becomes as follows.

Theorem 5.2. [12, Theorem 2.9] *For an abelian group G satisfying $1 \leq r_d(G) < \infty$, the following conditions are equivalent:*

- (i) G admits an SSGP topology;
- (ii) G admits an $\text{SSGP}(\alpha)$ topology for some ordinal α ;
- (iii) the quotient $H = G/t(G)$ of G with respect to its torsion part $t(G)$ has finite rank $r_0(H)$ and $r(H/A) = \omega$ for some (equivalently, every) free subgroup A of H such that H/A is torsion.

Corollary 4.3, Theorem 4.4 and Theorem 5.2 provide a complete solution to Question 1.8 in the abelian case.

Theorem 5.3. [12, Corollary 2.11] *For an abelian group G , the following conditions are equivalent:*

- (i) G admits an SSGP topology;
- (ii) G admits an $\text{SSGP}(\alpha)$ topology for some ordinal α .

Proof. By Remark 4.2, an abelian group G satisfies $r_d(G) = 0$ if and only if G is a bounded torsion group. Therefore, in case $r_d(G) = 0$, the conclusion of our theorem follows from Corollary 1.3 and (3). In case $1 \leq r_d(G) < \infty$, the result follows from the equivalence of items (i) and (ii) of Theorem 5.2. Finally, in case $r_d(G) \geq \omega$, the equivalence of items (i) and (ii) of our theorem follows from Theorem 4.4 and (3). \square

The next corollary provides a complete solution to Question 2.3 in the abelian case.

Corollary 5.4. [12] *For an abelian group G , the following conditions are equivalent:*

- (i) G admits an SSGP topology;
- (ii) G admits an $\text{SSGP}(n)$ topology for some $n \in \mathbb{N}$.

Proof. Since G is an abelian group, it follows from Theorem 3.3 that G is $\text{SSGP}(n)$ if and only if it is $\text{SSGP}(n)$. Now the conclusion follows from Theorem 5.3. \square

6. OUR RESULTS IN THE NON-COMMUTATIVE CASE

As far as the authors know, the non-commutative version of Question 1.8 remains completely open. Our next theorem positively resolves Question 1.8 in the case of free groups with infinitely many generators:

Theorem 6.1. *For an infinite set X , the free group $F(X)$ generated by X admits an SSGP topology. Furthermore, if X is countably infinite, then $F(X)$ admits a (separable) metric SSGP topology.*

The free group with a single generator is isomorphic to \mathbb{Z} , so it does not admit an SSGP topology by Example 1.5. This justifies the following

Question 6.2. Let $n \in \mathbb{N}$ and $n \geq 2$. Does the free group with n generators admit an SSGP topology?

In contrast with Theorem 6.1, another highly non-commutative group does not admit SSGP topologies.

Example 6.3. Let X be a set having at least two elements. Then the symmetric group $S(X)$ of all bijections of X with the composition of maps as group operation does not admit an SSGP topology. Indeed, suppose that \mathcal{T} is a (Hausdorff) group topology on $S(X)$. Assume first that X is finite. Then $S(X)$ is finite as well, so \mathcal{T} is discrete. Since $S(X)$ is non-trivial, \mathcal{T} cannot be SSGP. Suppose now that X is infinite. It is a well-known result of Gaughan that \mathcal{T} is stronger than the topology \mathcal{T}_p of pointwise convergence on $S(X)$, that is, the topology $S(X)$ inherits from the Tychonoff product X^X when X is equipped with the discrete topology [8]. Observe that $(S(X), \mathcal{T}_p)$ has many proper clopen subgroups. For example, $H = \{f \in S(X) : f(x_0) = x_0\}$, where $x_0 \in X$ is a fixed element, is such a subgroup. Since $\mathcal{T}_p \subseteq \mathcal{T}$, H is also \mathcal{T} -clopen. Finally, observe that a topological group containing a proper clopen subgroup cannot be SSGP.

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